

ON WITT'S THEOREM IN THE DENUMERABLY INFINITE CASE

BY
HERBERT GROSS

Introduction. Let E be a finite dimensional vectorspace over a field k , and let ϕ be a nondegenerate ε -hermitian form, $\phi: E \times E \rightarrow k$, (i.e. ϕ is a reflexive sesquilinear form, additive in both arguments and such that for $\lambda, \mu \in k$, $\phi(x\lambda, y) = \lambda^J \phi(xy)$, $\phi(x, y\mu) = \phi(x, y)\mu$ and $\phi(x, y) = \varepsilon \phi(y, x)^J$, where J is an involutory antiautomorphism and ε is in the center of k). If, in addition, ϕ is trace-valued (that is for every $x \in E$ there exists an $\alpha \in k$ such that $\phi(x, x) = \alpha + \varepsilon \alpha^J$) then Witt's famous theorem holds: if F is an arbitrary subspace of E then every metric homomorphism $\omega: F \rightarrow E$ which is injective can be extended to a metric automorphism of E [1]. Furthermore, provided F remains finite dimensional, the theorem holds for E of arbitrary dimension.

The following results concerning infinite subspaces of an infinite space E were proved in [4]. Let E be a k -vectorspace of denumerable dimension and ϕ a nondegenerate, alternate form⁽¹⁾ (i.e. $\phi(x, x) = 0$ for all $x \in E$). If F and $\bar{F}$ are subspaces with $F^\perp = (0)$, $\bar{F}^\perp = (0)$ and $\dim(E/F) = \dim(E/\bar{F})$ then there exists a metric automorphism of E which maps F onto $\bar{F}$. Furthermore, there also exists such an automorphism when F and $\bar{F}$ are closed subspaces ($F^{\perp\perp} = F$) with $F \subset F^\perp$, $\bar{F} \subset \bar{F}^\perp$, $\dim F = \dim \bar{F}$ and $\dim F^\perp = \dim \bar{F}^\perp$ (cf. the explicit normal form in 1.2 below).

In contrast to the finite case, however, prescribed isomorphisms between such subspaces F and $\bar{F}$ cannot, in general, be extended to automorphisms of E .

In the following we shall prove some isomorphism theorems of this kind, concerning both symmetric and alternate forms. They are generalizations of the results in [1] and [4]. The class of subspaces covered by these theorems includes in particular all the closed subspaces F and all the "dense" subspaces F . The dimension of E will be kept denumerably infinite throughout.

It is appropriate to mention that this line of attack follows closely Kaplansky's investigations in [4].

1. Notations and results.

1.1. Let E and $\bar{E}$ be k -vectorspaces, supplied with ε -hermitian forms ϕ and $\bar{\phi}$ respectively. A metric (orthogonal) isomorphism is a vector space isomorphism

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(1) All the forms discussed here are special instances of ε -hermitian forms.

$\varphi: E \rightarrow E$ such that $\bar{\varphi}(\varphi x, \varphi y) = \phi(x, y)$ for all $x, y \in E$. In particular, two subspaces F and $\bar{F}$ of E are said to be isomorphic ($F \cong \bar{F}$) if there is a metric isomorphism between F and $\bar{F}$ with respect to the restrictions $\phi|_F$ and $\phi|_{\bar{F}}$ of ϕ to F and $\bar{F}$ respectively. If F is a subspace of E then $F \cap F^\perp$ is called the radical ($\text{rad } F$) of F ; F is called semisimple if and only if its radical is trivial; ϕ is nondegenerate if and only if E itself is semisimple. As usual, we shall write $\|x\|$ for $\phi(x, x)$. A subspace F of E is said to be closed if $F^{\perp\perp} = F$. If F is closed and G is a finite dimensional subspace, then $F + G$ is closed. F is said to be dense in E , if $F^{\perp\perp} = E$, or $F^\perp = (0)$.

1.2. We now assume that ϕ is trace-valued and that E has denumerably infinite dimension (i.e. E possesses a denumerable algebraic basis). If F is a closed subspace of E with $F \subset F^\perp$ then there exists a closed subspace F' with $F' \subset F'^\perp$ and $F \cap F' = (0)$ such that $F \oplus F'$ has an orthogonal supplement in E , $E = (F \oplus F') \oplus (F \oplus F')^\perp$. Furthermore $F \oplus F'$ admits a basis $\{e_i, e'_i\}_{i \in I}$ with $\phi(e_i, e'_i) = \delta_{ij}$ and such that F is spanned by the vectors e_i and F' is spanned by the vectors e'_i , $i \in I$. In other words, $F \oplus F'$ is the orthogonal sum of card I hyperbolic planes $P_i = k(e_i, e'_i)$, ($\|e_i\| = \|e'_i\| = 0$, $\phi(e_i, e'_i) = 1$). Thus for a closed $F \subset E$ with $F \subset F^\perp$ we always have an orthogonal decomposition $E = (\bigoplus_I P_i) \oplus G$, $P_i = k(e_i, e'_i)$ a hyperbolic plane, $F = k[e_i]_{i \in I}$.

(A complete proof of this for the alternate case is given in [4]; it carries over without change, cf. [1, p. 78]).

1.3. Since we shall restrict ourselves to the symmetric case ($\phi(x, y) = \phi(y, x)$) and the alternate case ($\phi(x, x) = 0$), the antiautomorphism J (corresponding to ϕ) is necessarily the identity, hence the underlying field is commutative. If $\text{char } k \neq 2$, these forms are trivially trace-valued. When $\text{char } k = 2$, a trace-valued form is necessarily alternate.

1.4. In the case of a symmetric, nonalternate form we shall make further restrictions on the underlying field, and admit only Kneser fields. A Kneser field k is a commutative nonformally real field of characteristic $\neq 2$ with a finite group k^*/k^{*2} (the multiplicative group of nonzero elements modulo squares). For a detailed discussion of such fields see [3]. If ϕ is a nondegenerate symmetric form on the k -space E (E is always of denumerable dimension), and k is a Kneser field, then E possesses an orthonormal basis. In particular, every infinite dimensional subspace of E which is semisimple contains vectors x of arbitrarily prescribed length $\|x\|$. In fact, this is the crucial property we impose in order that the theorems proved below will hold in the symmetric case. The proofs hold without change for any such field. Fields other than Kneser fields for which the alleged property holds are for example the algebraic function fields in one variable over a finite constant field, as follows from the classical results on quadratic forms over such fields in the finite dimensional case (here the order of k^*/k^{*2} is $\aleph_0$). However our knowledge of examples other than Kneser fields is rather limited. The cases of some important formally real fields will be treated elsewhere.

1.5. In [2] the following theorem was proved: Let k be a Kneser field and ϕ a symmetric, nondegenerate form on E . Let F and $\bar{F}$ be subspaces of E with $F^\perp \subset F$, $\bar{F}^\perp \subset \bar{F}$ satisfying the conditions

$$F \cong \bar{F} \text{ and } \dim(E/F) = \dim(E/\bar{F}).$$

Then there exists an orthogonal isomorphism of E which maps F onto $\bar{F}$. This theorem will be essential in the proof of Theorem 2 below.

2. Isomorphism theorems.

LEMMA 1. *Let k be a Kneser field, E semisimple with respect to the symmetric form ϕ and of denumerable dimension. Let F and W be subspaces of E with the properties:*

- (1) F is a finite dimensional subspace spanned by the vectors $f_1, f_2, \dots, f_n$; and
- (2) W is an infinite dimensional semisimple subspace with $W^\perp \cap F = (0)$.

Then there exists a vector $x \in E$ such that

- (i) $x \in W$,
- (ii) $x \notin W \cap F$,
- (iii) $\|x\| = \alpha$ for arbitrarily prescribed $\alpha \in k$,
- (iv) $\phi(x, f_i) = \beta_i$ for arbitrarily prescribed $\beta_i \in k$ ($1 \leq i \leq n$).

Proof. (A) Let F_1 be the subspace of F spanned by $f_2, f_3, \dots, f_n$. By (2) we have $W^\perp \cap F_1 = (0)$. $F_1^\perp$ is trivially closed and of finite codimension in $F_1^\perp + W$. Therefore $F_1^\perp + W$ is closed and we find $F_1^\perp + W = (F_1^\perp + W)^{\perp\perp} = (F_1^{\perp\perp} \cap W^\perp)^\perp = (F_1 \cap W^\perp)^\perp = (0)^\perp = E$. Hence $\dim(W/(W \cap F_1^\perp)) = \dim((W + F_1^\perp)/F_1^\perp) = \dim(E/F_1^\perp) = \dim F_1$. Thus $W \cap F_1^\perp$ has an algebraic complement K in W , $K \oplus (W \cap F_1^\perp) = W$, with $\dim K = \dim F_1$.

(B) We next show that F is not perpendicular to $W \cap F_1^\perp$. For, if $F \subset (W \cap F_1^\perp)^\perp$ then trivially $F \cap K^\perp \subset (W \cap F_1^\perp)^\perp \cap K^\perp = [(W \cap F_1^\perp) + K]^\perp = W^\perp$. Since $F \cap W^\perp = (0)$ (by the assumption of the theorem), $F \cap K^\perp = (0)$. On the other hand, $\dim(F \cap K^\perp) \geq \dim F - \dim K = \dim F - \dim F_1 = 1$, a contradiction. Thus, F is not perpendicular to $W \cap F_1^\perp$. By the definition of F_1 we see therefore that f_1 , is not perpendicular to $W \cap F_1^\perp$. Hence there exists $x_1 \in W \cap F_1^\perp$ with

$$\phi(x_1, f_1) \neq 0, \phi(x_1, f_i) = 0 \quad (1 < i \leq n).$$

This procedure can be repeated n times with F_j , spanned by all the f_i with $i \neq j$, in lieu of F_1 . We find in this manner n vectors $x_1, x_2, \dots, x_n$ with $\phi(x_i, f_j) = 0$ for $i \neq j$ and $\phi(x_i, f_i) \neq 0$ ($1 \leq i, j \leq n$). The vector $z_0 = \sum_{i=1}^n \beta_i \phi(x_i, f_i)^{-1} \cdot x_i$ satisfies (i) and (iv).

(C) By 1.4 W possesses an orthonormal basis. Hence, by the same token $W \cap (F + k(z_0))^\perp$ contains an infinite dimensional subspace S spanned by an orthonormal basis. Therefore since k is not formally real, S contains a vector z_1 with $\|z_1\| = \alpha - \|z_0\|$ for arbitrarily prescribed $\alpha \in k$. The vector $z_2 = z_0 + z_1$

now possesses the properties (i), (iii) and (iv). Thus, if $z_2 \notin W \cap F$ then we are through with our construction. If we should have $z_2 \in W \cap F$, then we pick a vector s in S with $s \notin F$, $\|s\| = 0$ and $s \perp z_2$. The vectors $z_2 + s$ still has the properties (i), (iii) and (iv) and now we also have $z_2 + s \notin W \cap F$.

LEMMA 2. *Let k , (E, ϕ) and the subspace F be as in Lemma 1. Let W be an infinite dimensional subspace of E such that (1) $W^\perp = (0)$ and (2) $W + F \neq E$.*

Then there exists a vector $x \in E$ with the properties (iii) and (iv) of Lemma 1 and such that $x \notin W + F$.

Since $W^\perp = (0)$ the assumption $W^\perp \cap F = (0)$ of Lemma 1 is trivially satisfied and, by starting out with a vector z_0 with the properties (i) and (iv) (see Step (C)), a proof for Lemma 2 can be found. A complete proof is given in [2].

THEOREM 1. *Let k be a Kneser field, E semisimple with respect to the symmetric form ϕ and of denumerable dimension. Let V and $\bar{V}$ be infinite dimensional subspaces of E with the properties:*

- (1) $V^\perp$ and $\bar{V}^\perp$ are infinite dimensional and semisimple,
- (2) $\dim(V^{\perp\perp}/V) = \dim(\bar{V}^{\perp\perp}/\bar{V})$,
- (3) $\dim(E/(V^\perp + V^{\perp\perp})) = \dim(E/(\bar{V}^\perp + \bar{V}^{\perp\perp}))$.

Then there exists an orthogonal automorphism $T: E \rightarrow E$ with $T(V) = \bar{V}$.

Proof. We shall build the automorphism step by step. Assume that we have already constructed four finite dimensional subspaces $F_i, \bar{F}_i, G_i$ and $\bar{G}_i$ with the following properties:

- (a) $F_i \subset V^{\perp\perp}$, $\bar{F}_i \subset \bar{V}^{\perp\perp}$,
 - (b) $G_i \cap (V^\perp + V^{\perp\perp}) = G_i \cap V^\perp$, $\bar{G}_i \cap (\bar{V}^\perp + \bar{V}^{\perp\perp}) = \bar{G}_i \cap \bar{V}^\perp$.
- There exists an orthogonal isomorphism $T_i: F_i \oplus G_i \rightarrow \bar{F}_i \oplus \bar{G}_i$ with⁽²⁾
- (c) $T_i(F_i) = \bar{F}_i$,
 - (d) $T_i(F_i \cap V) = \bar{F}_i \cap \bar{V}$,
 - (e) $T_i(G_i) = \bar{G}_i$,
 - (f) $T_i(G_i \cap V^\perp) = \bar{G}_i \cap \bar{V}^\perp$.

We are going to enlarge $F_i \oplus G_i$ and $\bar{F}_i \oplus \bar{G}_i$ step by step in such a fashion that each basis vector e_n of a fixed basis $(e_n)_{n \geq 1}$ of E is eventually picked up by some $F_i \oplus G_i$ and some $\bar{F}_i \oplus \bar{G}_i$. In other words, we shall have

$$E = \bigcup_i F_i \oplus G_i = \bigcup_i \bar{F}_i \oplus \bar{G}_i.$$

Furthermore the construction will be done in such a fashion that at every step conditions (a) through (f) are satisfied: At every step the isomorphism T_i is extended to an orthogonal isomorphism $T_{i+1}: F_{i+1} \oplus G_{i+1} \rightarrow \bar{F}_{i+1} \oplus \bar{G}_{i+1}$ satisfying the properties (c) through (f) with $i+1$ in lieu of i . It is clear by properties (a) through (d), that in this fashion we obtain an orthogonal automorphism T of E which

(2) Notice that by the semisimplicity of $V^\perp$ conditions (a) and (b) imply that the sum $F_i + G_i$ is direct.

maps $V = \bigcup_i V \cap F_i$ onto $\bar{V} = \bigcup \bar{V} \cap \bar{F}_i$. We now describe one step in detail. Let e_n be the first basis vector not contained in $F_i \oplus G_i$. We will adjoin it to $F_i \oplus G_i$ in a suitable way and try to adjoin a matching vector x of E to $\bar{F}_i \oplus \bar{G}_i$.

Case 1. $e_n \notin (V^\perp \oplus V^{\perp\perp}) + (F_i \oplus G_i)$. Let us put $W = V^\perp \oplus V^{\perp\perp}$, $F = F_i \oplus G_i$ and similarly $\bar{W} = \bar{V}^\perp \oplus \bar{V}^{\perp\perp}$ and $\bar{F} = \bar{F}_i \oplus \bar{G}_i$. Thus $e_n \notin W + F$. In particular $W + F \neq E$. Therefore

$$(I) \quad \dim(F/W \cap F) = \dim(W + F/W) < \dim(E/W) = \dim(E/\bar{W}).$$

(The last equality follows from Hypothesis 3 of the theorem.) Furthermore, $(V^\perp \oplus V^{\perp\perp}) \cap (F_i \oplus G_i) = F_i + (V^\perp \cap G_i)$ and $(\bar{V}^\perp \oplus \bar{V}^{\perp\perp}) \cap (\bar{F}_i \oplus \bar{G}_i) = \bar{F}_i + (\bar{V}^\perp \cap \bar{G}_i)$ by (a) and (b), hence, by (c), (e) and (f), we obtain $\dim(F/(\bar{W} \cap F)) = \dim(F/(W \cap F))$. Therefore by using (I), $\dim((F + \bar{W})/\bar{W}) = \dim(F/(\bar{W} \cap F)) < \dim(E/\bar{W})$, from which we conclude that $F + \bar{W} \neq E$. Now let $f_1, \dots, f_n$ be a basis of F . By our induction assumption $T_i f_1, \dots, T_i f_n$ is then a basis for $\bar{F}$. By Lemma 2 there exists a vector $x \in E$ with the properties

$$(II) \quad x \notin \bar{W} + \bar{F}, \|x\| = \|e_n\|, \phi(x, T_i f_j) = \phi(e_n, f_j) \quad (1 \leq j \leq n).$$

We put $F_{i+1} = F_i$, $G_{i+1} = G_i \oplus k(e_n)$, $\bar{F}_{i+1} = \bar{F}_i$ and $\bar{G}_{i+1} = \bar{G}_i \oplus k(x)$. By virtue of (II) it is clear that T_i can be extended to an orthogonal isomorphism $T_{i+1}: F_{i+1} \oplus G_{i+1} \rightarrow \bar{F}_{i+1} \oplus \bar{G}_{i+1}$ by mapping e_n into x . Let

$$z \in G_{i+1} \cap (V^\perp \oplus V^{\perp\perp}),$$

then $z = \lambda e_n + g$, $g \in G_i$. Since $z \in V^\perp \oplus V^{\perp\perp}$ we have $\lambda e_n \in (V^\perp \oplus V^{\perp\perp}) + G_i$. Therefore $\lambda = 0$ since $e_n \notin (V^\perp \oplus V^{\perp\perp}) + (F_i \oplus G_i)$ in the present case. Thus $G_{i+1} \cap (V^\perp \oplus V^{\perp\perp}) = G_i \cap (V^\perp \oplus V^{\perp\perp})$. By the same reasoning $G_{i+1} \cap V^\perp = G_i \cap V^\perp$. Hence, together with (b), $G_{i+1} \cap (V^\perp \oplus V^{\perp\perp}) = G_i \cap (V^\perp \oplus V^{\perp\perp}) = G_i \cap V^\perp = G_{i+1} \cap V^\perp$. As x was chosen with $x \notin \bar{W} + \bar{F}$ we find by the same steps that $\bar{G}_{i+1} \cap (\bar{V}^\perp \oplus \bar{V}^{\perp\perp}) = \bar{G}_{i+1} \cap \bar{V}^\perp$. We see that the intersection property (b) again holds with $i+1$ in place of i . Since $G_{i+1} \cap V^\perp = G_i \cap V^\perp$ and $\bar{G}_{i+1} \cap \bar{V}^\perp = \bar{G}_i \cap \bar{V}^\perp$, the remaining conditions (a) through (f) remain trivially valid after the extension.

Case 2. $e_n \in (V^\perp \oplus V^{\perp\perp}) + (F_i \oplus G_i)$. We decompose e_n , $e_n = e'_n + e''_n + f + g$ with $e'_n \in V^\perp$, $e''_n \in V^{\perp\perp}$, $f \in F_i$, $g \in G_i$. In this case we shall adjoin e'_n and e''_n to G_i and F_i respectively in two successive steps. Assuming that e'_n is not already an element of G_i we first adjoin e'_n to G_i and try to enlarge $\bar{G}_i$ by a suitable match. By the semisimplicity of $\bar{V}^\perp$, $(\bar{V}^\perp \cap \bar{V}^{\perp\perp} = (0))$, and by (b) it follows that $\bar{V}^{\perp\perp} \cap \bar{G}_i = (0)$. Let $g_1, g_2, \dots, g_m$ be a basis of G_i . By (e), $T_i g_1, \dots, T_i g_m$ is a basis for $\bar{G}_i$. By Lemma 1 (with $\bar{V}^\perp$ and $\bar{G}_i$ in the roles of W and F respectively) there exists a vector $y \in E$ with the properties

$$(III) \quad y \in \bar{V}, y \notin \bar{V} \cap \bar{G}_i, \|y\| = \|e'_n\|, \phi(y, T_i g_j) = \phi(e'_n, g_j) \quad (1 \leq j \leq m)$$

Since $e'_n \in V^\perp$ and $y \in \bar{V}^\perp$ we have by (a) that $e'_n \perp F_i$ and $y \perp \bar{F}_i$. Using this together with (III) we can extend $T_i: F_i \oplus G_i \rightarrow \bar{F}_i \oplus \bar{G}_i$ to an orthogonal isomorphism $T_{i+1}: F_i \oplus G_i \oplus k(e'_n) \rightarrow \bar{F}_i \oplus \bar{G}_i \oplus k(y)$ simply by mapping e'_n into y . Setting $F_{i+1} = F_i$, $G_{i+1} = G_i + k(e'_n)$, $\bar{F}_{i+1} = \bar{F}_i$ and $\bar{G}_{i+1} = \bar{G}_i \oplus k(y)$ we find $G_{i+1} \cap (V^\perp \oplus V^{\perp\perp}) = G_i \cap (V^\perp \oplus V^{\perp\perp}) \oplus k(e'_n)$ and $G_{i+1} \cap V^\perp = (G_i \cap V^\perp) \oplus k(e'_n)$. By (b) we obtain again $G_{i+1} \cap (V^\perp \oplus V^{\perp\perp}) = G_{i+1} \cap V^\perp$ and similarly for $\bar{G}_{i+1}$. Furthermore, by the definition of the extension T_{i+1} we have $T_{i+1}(G_{i+1} \cap V^\perp) = T_i(G_i \cap V^\perp) + T_{i+1}(k(e'_n)) = (\bar{G}_i \cap \bar{V}^\perp) \oplus k(y) = \bar{G}_{i+1} \cap \bar{V}^\perp$. Thus, (b) and (f) remain valid after the extension. For the rest of the conditions this is trivially the case.

There remains e''_n to be adjoined to F_{i+1} . We recall that $e''_n \in V^{\perp\perp}$. Let us first consider the case where $e''_n \in V$ (assuming that e''_n is not in $F_i \oplus G_i$ already). Let H_1 be an algebraic complement of $G_{i+1} \cap V^\perp$ in G_{i+1} . Since $T_{i+1}(G_{i+1} \cap V^\perp) = \bar{G}_{i+1} \cap \bar{V}^\perp$ by (f), the image $\bar{H}_1 = T_{i+1}(H_1)$ is an algebraic complement of $\bar{G}_{i+1} \cap \bar{V}^\perp$ in $\bar{G}_{i+1}$. We next convince ourselves of the fact that

$$(IV) \quad V^\perp \cap (F_{i+1} \oplus H_1) = (0), \quad \bar{V}^\perp \cap (\bar{F}_{i+1} \oplus \bar{H}_1) = (0).$$

For, if $x \in V^\perp \cap (F_{i+1} \oplus H_1)$ with $x = f + g$ then $g \in V^\perp + F_{i+1} \subset V^\perp \oplus V^{\perp\perp}$ by (a). So by (b) $g \in (V^\perp \oplus V^{\perp\perp}) \cap G_{i+1} = V^\perp \cap G_{i+1}$. Consequently $g = 0$ since g also belongs to the complementary space H_1 . Furthermore $f = 0$ by (a) and the fact that $V^\perp$ is semisimple ($V^\perp \cap V^{\perp\perp} = (0)$).

By virtue of (IV) we may apply Lemma 1 (with $\bar{V}$ and $\bar{F}_{i+1} \oplus \bar{H}_1$ in the roles of W and F). We choose a basis $l_1, \dots, l_r$ of $F_{i+1} \oplus H_1$. $T_{i+1}l_1, \dots, T_{i+1}l_r$ is then a basis for $\bar{F}_{i+1} \oplus \bar{H}_1$. There exists a vector $z \in E$ with the properties

$$(V) \quad z \in \bar{V}, z \notin \bar{V} \cap (\bar{F}_{i+1} \oplus \bar{H}_1), \|z\| = \|e''_n\|, \phi(z, T_{i+1}l_j) = \phi(e''_n, l_j) \quad (1 \leq j \leq r)$$

We set $F_{i+2} = F_{i+1} \oplus k(e''_n)$, $G_{i+2} = G_{i+1}$, $\bar{F}_{i+2} = \bar{F}_{i+1} \oplus k(z)$ and $\bar{G}_{i+2} = \bar{G}_{i+1}$. We have

$$(VI) \quad G_{i+1} = (G_{i+1} \cap V^\perp) \oplus H_1 \quad \text{and} \quad \bar{G}_{i+1} = (\bar{G}_{i+1} \cap \bar{V}^\perp) \oplus \bar{H}_1.$$

Furthermore by (a) we have $e''_n \perp G_{i+1} \cap V^\perp$ and $z \perp \bar{G}_{i+1} \cap \bar{V}^\perp$. Thus, by (V) and (VI) we extend T_{i+1} to an orthogonal isomorphism $T_{i+2}: F_{i+2} \oplus G_{i+2} \rightarrow \bar{F}_{i+2} \oplus \bar{G}_{i+2}$ by mapping e''_n into z . Since $e''_n \in V$ and $z \in V$ we find $F_{i+2} \cap V = (F_{i+1} \cap V) + k(e''_n)$ etc. and conditions (a) through (f) remain trivially valid after the extension. The case where $e''_n \in V^{\perp\perp}$ but $e''_n \notin V$ remains to be discussed. We have $V + F_{i+1} \neq V^{\perp\perp}$ since $e''_n \notin V + F_{i+1}$ by assumption. Therefore,

$$\dim(F_{i+1}/(V \cap F_{i+1})) = \dim((F_{i+1} + V)/V) < \dim(V^{\perp\perp}/V) = \dim(\bar{V}^{\perp\perp}/\bar{V})$$

where the last equality is obtained by the assumption 2 of the theorem. By (c) and (d) we obtain $\dim(F_{i+1}/(V \cap F_{i+1})) = \dim(\bar{F}_{i+1}/(\bar{V} \cap \bar{F}_{i+1}))$ hence

$$\dim((F_{i+1} + \bar{V})/\bar{V}) = \dim(\bar{F}_{i+1}/(\bar{V} \cap \bar{F}_{i+1})) < \dim(\bar{V}^{\perp\perp}/\bar{V})$$

and we learn that $\bar{V} + F_{i+1} \neq \bar{V}^{\perp\perp}$. Let $\bar{V}^{\perp_0}$ be the orthogonal complement of $\bar{V}$ in $\bar{V}^{\perp\perp}$. Since $\bar{V}^{\perp}$ is semisimple, we have $\bar{V}^{\perp_0} = (0)$. Hence we can apply Lemma 2 to the semisimple vectorspace $(\bar{V}^{\perp\perp}, \phi|_{\bar{V}^{\perp\perp}})$, $\bar{V}$ and F_{i+1} in the roles of W and F respectively. We choose a basis $f_1, \dots, f_m$ for F_{i+1} and find a vector $z_0 \in \bar{V}^{\perp\perp}$ with the properties

$$(VII) \quad z_0 \notin \bar{V} + F_{i+1}, \|z_0\| = \|e_n''\| \text{ and } \phi(z_0, T_{i+1}f_j) = \phi(e_n'', f_j) \quad (1 \leq j \leq m).$$

Then let $g_1, \dots, g_s$ be a basis of H_1 where H_1 is again an algebraic complement of $G_{i+1} \cap V^{\perp}$ in G_{i+1} . $T_{i+1}g_1, \dots, T_{i+1}g_s$ is then the basis of an algebraic complement $\bar{H}_1$ of $\bar{G}_{i+1} \cap \bar{V}^{\perp}$ in $\bar{G}_{i+1}$. One proves as before that

$$\bar{V}^{\perp} \cap (F_{i+1} \oplus \bar{H}_1 \oplus k(z_0)) = (0).$$

Once more we apply Lemma 1 (with $\bar{V}$ and $F_{i+1} \oplus \bar{H}_1 \oplus k(z_0)$ in the roles of W and F respectively) and find a vector $z_{00} \in E$ with

$$(VIII) \quad \begin{aligned} z_{00} \in \bar{V}, z_{00} \notin \bar{V} \cap F_{i+1} \oplus \bar{H}_1 \oplus k(z_0)^{(3)}, \|z_{00}\| = 0. \\ \phi(z_{00}, z_0) = 0, \phi(z_{00}, T_{i+1}f_j) = 0, \phi(z_{00}, T_{i+1}g_l) = \phi(e_n'', g_l) - \phi(z_0, T_{i+1}g_l) \\ (1 \leq j \leq m, 1 \leq l \leq s). \end{aligned}$$

We set $z = z_0 + z_{00}$ and by (VII) and (VIII) we find that

$$(IX) \quad \begin{aligned} z \in \bar{V}^{\perp\perp}, z \notin \bar{V} + F_{i+1} + \bar{G}_{i+1}, \|z\| = \|e_n''\|, \\ \phi(z, T_{i+1}f_j) = \phi(e_n'', f_j) \text{ and } \phi(z, T_{i+1}g_l) = \phi(e_n'', g_l). \end{aligned}$$

We put $F_{i+2} = F_{i+1} \oplus k(e_n'')$, $G_{i+2} = G_{i+1}$, $\bar{F}_{i+2} = \bar{F}_{i+1} \oplus k(z)$ and $\bar{G}_{i+2} = \bar{G}_{i+1}$. As before we have $e_n'' \perp G_{i+1} \cap V^{\perp}$, $z \perp \bar{G}_{i+1} \cap \bar{V}^{\perp}$.

From (IX) we see that T_{i+1} can again be extended to $T_{i+2}: F_{i+2} \oplus G_{i+2} \rightarrow F_{i+2} \oplus \bar{G}_{i+2}$. In this case we have $F_{i+2} \cap V = F_{i+1} \cap V$ and $\bar{F}_{i+2} \cap \bar{V} = \bar{F}_{i+1} \cap \bar{V}$. All the conditions (a) through (f) remain trivially satisfied after this extension.

This completes the proof of our theorem. Summarizing, we remark that we have seen how to enlarge $F_i \oplus G_i$ and $\bar{F}_i \oplus \bar{G}_i$ by at least one dimension in such a fashion that (a) through (f) remain valid after the extension. Furthermore, $F_i \oplus G_i$ has been enlarged by adjunction of a prescribed basis vector. It is clear that in this fashion we can arrive at $\bigcup_i F_i \oplus G_i = E$. However, in order to make sure that $\bigcup_i \bar{F}_i \oplus \bar{G}_i = E$ also, one has to alternate between $F_i \oplus G_i$ and $\bar{F}_i \oplus \bar{G}_i$. In the next step the first basis vector not contained in $\bar{F}_{i+1} \oplus \bar{G}_{i+1}$ is added to $\bar{F}_{i+1} \oplus \bar{G}_{i+1}$ and $F_{i+1} \oplus G_{i+1}$ this time is enlarged by a suitable match.

We are now in a position to prove our main theorem. It covers the class of subspaces $H \subset E$ for which $\text{rad } H = \text{rad } (H^{\perp})$. In particular, this condition is satisfied by closed subspaces. Examples of spaces which are not closed and which satisfy this condition are the dense subspaces since every space H with $H^{\perp} \subset H$ falls into our category.

(3) This property will not be needed.

THEOREM 2. Let k be a Kneser field⁽⁴⁾, E semisimple with respect to the symmetric form ϕ and of denumerable dimension. Let H and $\bar{H}$ be isomorphic subspaces with $\text{rad } H = \text{rad}(H^\perp)$ and $\text{rad } \bar{H} = \text{rad}(\bar{H}^\perp)$ satisfying the following properties:

- (i) $H^\perp \cong \bar{H}^\perp$,
- (ii) $\dim((H^\perp + H^{\perp\perp})/(H^\perp + H)) = \dim((\bar{H}^\perp + \bar{H}^{\perp\perp})/(\bar{H}^\perp + \bar{H}))$,
- (iii) $\dim((\text{rad } H)^\perp/(H^\perp + H^{\perp\perp})) = \dim((\text{rad } \bar{H})^\perp/(\bar{H}^\perp + \bar{H}^{\perp\perp}))$.

Then there exists an orthogonal automorphism of E which maps H onto $\bar{H}$.

Conditions (i), (ii) and (iii) are trivially necessary for such an automorphism to exist. Furthermore, condition (i) can be replaced (under the present conditions) by simpler conditions on dimensions, namely

$$(i') \dim((H + H^\perp)/H) = \dim((\bar{H} + \bar{H}^\perp)/\bar{H})$$

provided that this dimension is infinite. If $\dim((H + H^\perp)/H)$ is finite, the (necessary) condition that $H^\perp$ and $\bar{H}^\perp$ be isomorphic is not implied by (i') and the other conditions, unless k is "perfect" (i.e. the order of k^*/k^{*2} is 1). For otherwise, there are at least two nonisomorphic 1-dimensional k -spaces L_1 and L_2 . The external sums $E \oplus L_1$ and $E \oplus L_2$ possess orthonormal bases, hence $E \oplus L_1 \cong E \oplus L_2$ and therefore they provide a simple counterexample.

Proof. (a) Let H_0 be an algebraic complement of $\text{rad } H$ in H and H_1 an algebraic complement of $\text{rad } H$ in $H^\perp$. We then have the orthogonal decomposition $H + H^\perp = H_0 \oplus \text{rad}(H) \oplus H_1$. Furthermore let L be an algebraic complement of H in $H^{\perp\perp}$. Since $\text{rad } H = \text{rad}(H^\perp)$, we find that $(H + H^\perp) \cap H^{\perp\perp} = H$. In particular, $L \cap H = (0)$ and we have the direct decomposition

$$H + H^\perp + H^{\perp\perp} = H_0 \oplus \text{rad}(H) \oplus H_1 \oplus L.$$

(b) Since $\text{rad } H = \text{rad}(H^\perp) = (H + H^\perp)^\perp$ we see that $\text{rad } H$ is a closed space. Therefore by 1.2 there exists a subspace R with $R \cap \text{rad } H = (0)$ and a decomposition $E = (\text{rad}(H) \oplus R) \oplus E_0$ with $E_0 \perp \text{rad}(H) + R$, where $\text{rad}(H) \oplus R$ is an orthogonal sum of hyperbolic planes:

$$\text{rad}(H) \oplus R = \bigoplus_{i \in I} P_i,$$

$$(1) \quad \text{card } I = \dim(\text{rad } H),$$

$$P_i = k(e_i, e'_i) \text{ with } \text{rad } H = k[e_i]_{i \in I}, R = k[e'_i]_{i \in I}$$

There exist subspaces H_0^*, H_1^*, L^* all orthogonal to R such that

$$\text{rad}(H) \oplus H_0^* = \text{rad}(H) \oplus H_0 (= H),$$

$$\text{rad}(H) \oplus H_1^* = \text{rad}(H) \oplus H_1 (= H^\perp),$$

$$\text{rad}(H) \oplus L^* = \text{rad}(H) \oplus L \text{ (so } \text{rad}(H) \oplus H_0^* \oplus L^* = H^{\perp\perp}).$$

(4) Cf. the discussion at the end of 1.4.

Since $\text{rad } H \perp H_0^*, H_1^*, L^*$ we have $H_0^*, H_1^*, L^* \subset E_0$. Without loss of generality we may assume that H_0^* was chosen for H_0 at the beginning and similarly for H_1^* and L_1^* . Hence we shall drop the asterisks. Thus, if K is some algebraic complement of $H_0 \oplus H_1 \oplus L (= H_0^* \oplus H_1^* \oplus L_1^*)$ in E_0 we have the decompositions

$$(2) \quad E = (\text{rad}(H) \oplus R) \oplus E_0 \text{ with } E_0 \perp (\text{rad}(H) + R) \text{ and } E_0 = H_0 \oplus H_1 \oplus L \oplus K.$$

We find $(\text{rad } H)^\perp = \text{rad}(H) \oplus E_0$. This gives

$$(3) \quad \dim K = \dim((\text{rad } H)^\perp / (H^\perp + H^{\perp\perp})).$$

We also read off

$$(4) \quad \dim L = \dim((H^\perp + H^{\perp\perp}) / (H^\perp + H)).$$

With respect to the subspace $\bar{H}$ we have a similar decomposition

$$(5) \quad \begin{aligned} E &= (\text{rad}(\bar{H}) \oplus \bar{R}) \oplus \bar{E}_0 \text{ with } \bar{E}_0 \perp (\text{rad}(\bar{H}) \oplus \bar{R}) \text{ and } \text{rad}(\bar{H}) \oplus \bar{R} = \bigoplus_{i \in \bar{I}} \bar{P}_i, \\ \bar{E}_0 &= \bar{H}_0 \oplus \bar{H}_1 \oplus \bar{L} \oplus \bar{K} \end{aligned}$$

(with $\bar{P}_i, \bar{H}_0, \bar{H}_1, \bar{L}, \bar{K}$ analogous objects to those above).

(c) By assumption H and $\bar{H}$ are isomorphic, in particular $\dim(\text{rad } \bar{H}) = \dim(\text{rad } H)$ since a radical is mapped on a radical under any isomorphism. We therefore have the same number of hyperbolic planes in the decompositions (1) and (5) of $\text{rad}(H) \oplus R$ and $\text{rad}(\bar{H}) \oplus \bar{R}$ respectively: $\text{card } I = \text{card } \bar{I}$. Hence, by linear extension of the index map which sends the basis vectors e_i, e'_i of P_i into the basis vectors $\bar{e}_i, \bar{e}'_i$ of $\bar{P}_i$ respectively, we obtain an orthogonal isomorphism

$$(6) \quad T_1: \text{rad}(H) \oplus R \rightarrow \text{rad}(\bar{H}) \oplus \bar{R} \text{ with } T_1(\text{rad } H) = \text{rad } \bar{H}.$$

Similarly, $H \cong \bar{H}$ implies that any two algebraic complements of (the totally isotropic spaces) $\text{rad } H$ and $\text{rad } \bar{H}$ in H and $\bar{H}$ respectively, are isomorphic. In particular

$$(7) \quad H_0 \cong \bar{H}_0.$$

Since furthermore $H^\perp \cong \bar{H}^\perp$ by assumption of the theorem and since $\text{rad}(H^\perp) = \text{rad } H$ etc., by the same token,

$$(8) \quad H_1 \cong \bar{H}_1.$$

By assumptions (ii) and (iii) of the theorem and by (3) and (4) (and their analogues for $\bar{L}$ and $\bar{K}$) we obtain

$$(9) \quad \dim L = \dim \bar{L} \text{ and } \dim K = \dim \bar{K}.$$

(d) We now consider various cases which may arise and which have to be treated separately.

Case 1. H_0 and H_1 are both finite dimensional. Since we have seen that

$\text{rad } H$ is closed, the same is true for $H = \text{rad}(H) \oplus H_0$ and hence for $H^{\perp\perp} + H^\perp = H + H^\perp = H \oplus H_1$ (1.1). Therefore $H^\perp + H^{\perp\perp} = (H^\perp + H^{\perp\perp})^{\perp\perp} = (H^{\perp\perp} \cap H^\perp)^\perp = (H \cap H)^\perp = (\text{rad } H)^\perp$. Hence $L = K = \bar{L} = \bar{K} = (0)$ by (3), (4) and (9). E decomposes as follows:

$$E = (\text{rad}(H) \oplus R) \oplus (H_0 \oplus H_1) \text{ and } E = (\text{rad}(\bar{H}) \oplus \bar{R}) \oplus (\bar{H}_0 \oplus \bar{H}_1).$$

Since $H_0 \oplus H_1$ and $\bar{H}_0 \oplus \bar{H}_1$ are orthogonal summands in these decompositions, isomorphisms (6), (7) and (8) define an automorphism of E which maps $H = \text{rad}(H) \oplus H_0$ on $\bar{H} = \text{rad}(\bar{H}) \oplus \bar{H}$ ($H_0 \perp H_1, \bar{H}_0 \perp \bar{H}_1$!).

We are left with the case where at least one of H_0 and H_1 is infinite dimensional. In particular, the semisimple spaces E_0 and $\bar{E}_0$ are then both infinite dimensional and so possess orthonormal bases (1.4). In other words, we have

$$E_0 \cong \bar{E}_0.$$

Case 2. H_0 is finite dimensional. Since H_0 is also semisimple, H_0 has an orthogonal supplement E_1 in E_0 . The same is the case for $\bar{H}_0$ by virtue of (7). So

$$E_0 = H_0 \oplus E_1, \bar{E}_0 = \bar{H}_0 \oplus \bar{E}_1 \text{ with } E_1 \perp H_0, \bar{E}_1 \perp \bar{H}_0.$$

E_1 and $\bar{E}_1$ are semisimple and they are infinite dimensional since they contain the infinite dimensional spaces H_1 and $\bar{H}_1$ respectively. Thus E_1 and $\bar{E}_1$ possess orthonormal bases and therefore

$$E_1 \cong \bar{E}_1.$$

E decomposes into orthogonal subspaces as follows

$$E = (\text{rad}(H) \oplus R) \oplus H_0 \oplus E_1, E = (\text{rad}(\bar{H}) \oplus \bar{R}) \oplus \bar{H}_0 \oplus \bar{E}_1$$

and this case is now completed as in case 1.

Case 3. H_1 is finite dimensional. E decomposes into orthogonal subspaces as in the second case:

$$(10) \quad E = (\text{rad}(H) \oplus R) \oplus H_1 \oplus E_2, E = (\text{rad}(\bar{H}) \oplus \bar{R}) \oplus \bar{H}_1 \oplus \bar{E}_2, E_2 \cong \bar{E}_2$$

Let F ($\bar{F}$) be an algebraic complement of H_0 ($\bar{H}_0$) in E_2 ($\bar{E}_2$). If $H_0^{\perp_0}$ ($\bar{H}_0^{\perp_0}$) is the orthogonal complement of H_0 ($\bar{H}_0$) in E_2 ($\bar{E}_2$) we find that

$$(11) \quad H_0^{\perp_0} = (0) \text{ and } \bar{H}_0^{\perp_0} = (0).$$

Furthermore, $\dim F = \dim K + \dim L$ and $\dim \bar{F} = \dim \bar{K} + \dim \bar{L}$. By (9) therefore

$$(12) \quad \dim(E_2/H_0) = \dim(\bar{E}_2/\bar{H}_0).$$

In view of (11) and (12) we can apply the theorem quoted in 1.5: There exists an orthogonal isomorphism $T_0: E_2 \rightarrow \bar{E}_2$ with $T_0(H_0) = \bar{H}_0$. From isomorphisms (6) and (8) and decomposition (10) we thus secure an orthogonal automorphism of E which maps $H = \text{rad}(H) \oplus H_0$ onto $\bar{H} = \text{rad}(\bar{H}) \oplus \bar{H}_0$.

We are thus left with

Case 4. H_0 and H_1 are both infinite dimensional. We have

$$(13) \quad E_0 = H_0 \oplus H_1 \oplus L \oplus K, \bar{E}_0 = \bar{H}_0 \oplus \bar{H}_1 \oplus \bar{L} \oplus \bar{K} \text{ and } E_0 \cong \bar{E}_0.$$

In order to find an isomorphism $T_0: E_0 \rightarrow \bar{E}_0$ which maps H_0 onto $\bar{H}_0$ we shall apply Theorem 1 to the space E_0 . The orthogonal complement $H_0^{\perp_0}$ of H_0 in E_0 is precisely H_1 , similarly $\bar{H}_0^{\perp_0} = \bar{H}_1$. Finally $H_1^{\perp_0} = H_0 \oplus L$ and $\bar{H}_1^{\perp_0} = \bar{H}_0 \oplus \bar{L}$. Hence

$$\dim(H_0^{\perp_0 \perp_0} / H_0) = \dim L = \dim \bar{L} = \dim(\bar{H}_0^{\perp_0 \perp_0} / \bar{H}_0)$$

and

$$\dim(E_0 / (H_0^{\perp_0} + H_0^{\perp_0 \perp_0})) = \dim K = \dim \bar{K} = \dim(\bar{E}_0 / (\bar{H}_0^{\perp_0} + \bar{H}_0^{\perp_0 \perp_0})).$$

Finally we recall that H_0 and H_1 are both trivially semisimple. Thus the assumptions of Theorem 2 are satisfied: there exists an orthogonal isomorphism $T_0: E_0 \rightarrow \bar{E}_0$ with $T_0(H_0) = \bar{H}_0$. Together with the isomorphism T_1 of (b) we obtain an orthogonal isomorphism of E which maps $H = \text{rad}(H) \oplus H_0$ onto $\bar{H} = \text{rad}(\bar{H}) \oplus \bar{H}_0$. This completes the proof of Theorem 2.

If ϕ is a nondegenerate alternate form on the infinite dimensional space E (1.3) then Lemmas 1 and 2, with the obvious modification that (iii) be dropped, hold for any commutative field whatsoever. For this reason the theorem quoted in 1.5 carries over to the alternate case for any arbitrary underlying field and the proof of Theorem 2 goes through mutatis mutandis (for example, $E_0 \cong \bar{E}_0$ in the last three cases because these spaces now possess symplectic bases and are of the same dimension). Furthermore, conditions on isomorphisms can now be replaced by simpler conditions on dimensions, as every semisimple space of at most denumerable dimension now admits of a symplectic basis. Thus, we obtain the following companion to Theorem 2:

THEOREM 3. *Let ϕ be a nondegenerate, alternate form on the k -vectorspace E of denumerable dimension, and k an arbitrary commutative field (of any characteristic). Let H and $\bar{H}$ be subspaces of E with $\text{rad}(H^\perp) = \text{rad } H$, $\text{rad}(\bar{H}^\perp) = \text{rad}(\bar{H})$ and satisfying the following properties:*

- (i) $\dim H = \dim \bar{H}$, $\dim(\text{rad } H) = \dim(\text{rad } \bar{H})$,
- (ii) $\dim(H / \text{rad } H) = \dim(\bar{H} / \text{rad } \bar{H})$,
- (iii) $\dim(H + H^\perp / H) = \dim((\bar{H} + \bar{H}^\perp) / \bar{H})$,
- (iv) $\dim((H^\perp + H^{\perp \perp}) / (H^\perp + H)) = \dim((\bar{H}^\perp + \bar{H}^{\perp \perp}) / (\bar{H}^\perp + \bar{H}))$,
- (v) $\dim((\text{rad } H)^\perp / (H^\perp + H^{\perp \perp})) = \dim((\text{rad } \bar{H})^\perp / (\bar{H}^\perp + \bar{H}^{\perp \perp}))$.

Then there exists a metric automorphism of E which maps H onto $\bar{H}$.

The last theorem partly confirms a conjecture by Kaplansky to the effect that equality of corresponding cardinal numbers defined by the subspaces H and $\bar{H}$ is sufficient in the alternate case for an automorphism of the required kind to exist [4].

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MONTANA STATE UNIVERSITY,
BOZEMAN, MONTANA